Stochastic Dynamical Systems Modeling Ecosystems Stability

Research Thesis

Presented in partial fulfillment of the requirements for graduation with research distinction in physics in the undergraduate colleges of The Ohio State University

by

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I. Introduction

The typical state of a natural ecosystem is such that the variables that characterize it like vegetation and quality of water fluctuate in time around an average value. However there are events that lead to regime shifts on very small time scales. Unsurprisingly, large scale events such as hurricanes can trigger this but small changes of a parameter within a system can also lead to this catastrophic shift (Scheffer, Carpenter 2003). This has been attributed to the proximity of the system to a threshold in a parameter value that separates different regimes. The idea is that the system has multiple, sometimes coexisting, steady states one of which is usually ecologically desirable and the other degraded. Theoretical approaches have relied on simple differential equations to describe the time evolution of these complex systems. It is well-known from the theory of non-linear ordinary differential equationss that multiple steady states can occur (as we will see explicitly in this thesis) and the number and qualitative nature of steady states can change as a function of the parameters at isolated values of the parameters in nonlinear differential equations called bifurcation points. This phenomenon was investigated by May in 1977, where he examined a simple ODE that described the vegetation or a biomass in a semi-arid system that shows two steady states one of which can be described as bare and the other vegetated. (May 1977). The equation he studied was

$$\frac{\partial V}{\partial t} = rV(1 - \frac{V}{V_c}) - c\frac{V^2}{V^2 + V_0^2} \qquad (1)$$

This equation is whose behavior we will study in this thesis under spatial and stochastic effects.

i. Environmental Parameters

This section discusses the origins of the above equation as originally justified. A grazing ecosystem is one that consists of a population of her-

bivores which is maintained at some constant density ρ which is sustained by vegetation whose biomass is V. The most important factor to track here is vegetation biomass. If we assume that there is no grazing, then the growth rate of vegetation as a function of V is $G(V)$. Now, if the herbivores consumption rate is $C(V)$, Then the rate that V changes by is

$$\frac{\partial V}{\partial t} = G(V) - C(V)$$

These two competing terms imply that if the growth rate and the consumption rate are equal, we have a fixed point. $G(V)$ is given by the logistic equation $G(V) = rV(1 - V/k)$ and $C(V) = \beta\rho V^2/(V_0^2 + V^2)$. We define $c = \beta\rho$, which leaves us with

$$\frac{\partial V}{\partial t} = rV(1 - V/V_c) - cV^2/(V_0^2 + V^2) \quad (2)$$

Note the properties of the equations for $C(V)$ and $G(V)$. $G(v)$ has zeros $V = 0$ and $V = V_c$. The equation for $C(V)$ only has one zero at $V = 0$, but the more interesting feature of $C(V)$ is how it acts for small and large V. For large V, it is essentially c and for small V, it shrinks c by a factor of $1 + V_0^2/V^2$. Now focusing on parameters in this system, specifically r and c, we can naively think of these as rate of growth and rate of decay respectively. r can represent ecological quantities such as rainfall and and soil quality, while c can represent grazing herbivores, human exhaustion of resources, or some ecological pathogen. Given the behavior of $C(V)$ the decay rate has a max value of c if the area is very vegetated and a min value of $c/(1 + V_0^2/V^2)$ if there is very little vegetation. The next question is what are the steady states of this equation. We note that this equation is not linear in V so for different values of parameters we may have 3 real solutions.

ii. Bifurcation

We note that the simplified description ignores variation of the vegetation V with space and so we have an ODE. We illustrate the phenomenon of bifurcation by examining a much simpler ODE.

$$\dot{x} = \mu x - x^3$$

We examine the stability of fixed points by making a small perturbation $x(t) = x^* + \epsilon(t)$ where $x*$ is a fixed point and $<< x*$ is a small perturbation, So we obtain a linear ODE to see if the perturbation grows or decays. This is equivalent to linear stability analysis. We note that if $\mu < 0$, then there is only one steady state at $x = 0$. When $\mu > 0$ we then have three steady states: $x = 0, \pm\sqrt{\mu}$. We also have a stable solution when $\mu > 0$ for $x = 0$ and unstable when $\mu < 0$. This is a clear showing that there are multiple steady states available to a differential equation of our kind.

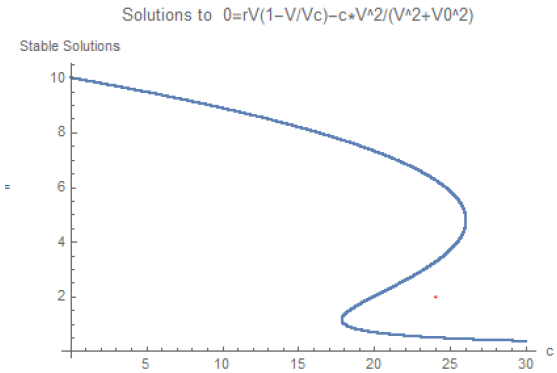
This has large consequences in our system, as described by May (May 1977). If a high vegetated stable steady state exists, which it does, and a low vegetative stable steady state also exists with V much less than the high vegetated state, then our goal is to identify properties about stability between these two states or phases and try to find ways to promote the highly vegetated state.

We are interested in how the asymptotic (i.e. long-time or steady state) behavior changes in our system when we vary parameters in this system. The parameter we are interested in varying is c since it can be thought of as consumption rate and we call this the bifurcation parameter. We want to find states in which our equation does not change in time when $\frac{dV}{dt} = 0$ these are called fixed points. We find all fixed points for reasonable choices of c. The fixed point values of the vegetation V as a function of the bifurcation parameter c are displayed in Figure 1. It shows coexisting fixed points in the range of c (18-26) .

Now that we have identified the fixed points we are now interested in their stability as dependent on c. With a quick calculation, we find that there are two sections that range with c of fixed points where a small perturbation will settle back to the fixed point. The two stable sections of fixed points are what we call the high vegetated state, which corresponds to the the highly vegetated section of the graph at low c

and the low vegetated state at high c. The fixed points that are between the high vegetation and the low vegetation are unstable, so a small perturbation added to it will push it towards either the high vegetated state or low vegetated state. Proving that these two steady states exist allows us to ask questions such as which state is more stable and does the stability depend on the bifurcation parameter c?

Figure 1: A bifurcation diagram describing the general stability created by our environmental parameters. If we start with an initial condition of $V = 2$ and $c = 24$ we end up at the red dot produced below. This red dot will head down towards the less favorable state.



iii. Spatial Dependence

After May's initial investigation about average values, people were interested in also applying seed dispersal effects in systems like these. There have been complicated ways of adding this effect in pursuit of realism and long distance seed dispersal before and these have been seen to be extremely close to a simple near neighbor distribution (Guttal, Jayaprakash 2008). For our system to simulate seed dispersal we add a diffusive term so our equation becomes

$$\frac{\partial V}{\partial t} = D\nabla^2 V + rV(1 - \frac{V}{V_c}) - c\frac{V^2}{V^2 + V_0^2} \quad (3)$$

Where V is vegetation biomass, t is time, D is the diffusion constant, r is growth rate, V_c is carrying capacity under a mean grazing rate, and c is the mean grazing rate.

Let us examine the behavior of our system for a value of c where multiple steady states exist: the bare and highly vegetated states are separated by an unstable fixed point. For a given value of c if the initial state has a vegetation V value above or below the unstable fixed point value under time evolution the system flows to the vegetated or the bare fixed points respectively. This is the prediction of the mean-field model. We will see from our simulations (and those done before), that in a spatially explicit model the average value can be in the domain of attraction of the vegetated fixed point; however, depending on the detailed spatial profile the system can become bare. This emphasizes the need for spatially extended models.

II. Simulating the Lattice

Studying the time evolution of the partial differential equation (3) with stochastic noise in two dimensions and characterizing its behavior is only possible using numerical methods. Fortunately, this is possible on a personal laptop with the algorithm described below implemented in C++. The equation is solved numerically using an Euler schema. The update scheme is shown below

$$V[i,j,t+\Delta t] = \Delta t\Bigg(\frac{D}{\Delta x^2}\Big(V[i+\Delta x,j,t] + V[i-\Delta x,j,t] + V[i,j+\Delta x,t] + V[i,j-\Delta x,t] - 4V[i,j,t]\Big) + r[i,j]V[i,j,t](1 - \frac{V[i,j,t]}{V_c}) - c[i,j]\frac{V[i,j,t]^2}{V[i,j,t]^2 + V_0[i,j,t]^2}\Bigg) + V[i,j,t]$$

Where Δt and Δx are the time step and spatial step for each iteration. For our lattice, the space between each site, or each Δx, is given to be $10m$ and Δt is .001 years. Looking at table

1, we can see all the parameters with the value used in the program, how that value is related to the units of Δt and Δx, and the value of σ used to simulate stochastic noise.

The simulations were done in one and two dimensions on lattices of size N and $N \times N$ respectively and run on timescales 1-80 years.

III. 1-D

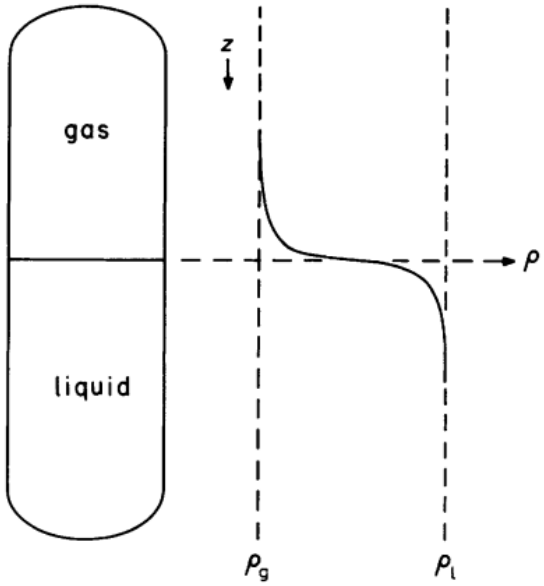
i. Interface Velocity No Noise

A simple set up to study spatially extended models is a liquid-vapour equilibrium across a planar interface in a one dimensional fluid. We treat molecules as hard spheres which have a potential associated with them and ignore any correlation other then hard sphere exclusion. We see in figure 2 the liquid vapor case which creates an interface. We are interested in how this interface moves to measure relative stability of our different phases. We will follow this approach in this ecological model. The simplest interface is a line interface (no curvature) in two dimensions. This has the advantage that it can be analyzed analytically and simulated numerically easily and we report these results first. Then we discuss the two-dimensional case. We begin by discussing our one dimensional system without stochastic noise. This is governed by the following equation.

$$\frac{\partial V(x,t)}{\partial t} = D\nabla^2 V(x,t) + rV(x,t)(1 - \frac{V(x,t)}{V_c}) - c\frac{V(x,t)^2}{V(x,t)^2 + V_0^2}$$

This one dimensional investigation is important because it was a model that has been proposed to accurately portray ecosystems so we can gain insight in stability from this (May 1977). We then want to investigate it further for two reasons. One to investigate new phenomenons within this 1-D system and two to have a baseline to compare our 2-D model against. Our first investigation was into the relative stability of the steady states that this equation has available to it. Namely, the high vegetated state and the low vegetated state. To do this, we choose to split the lattice in half such that half has initial conditions that lead to the lower steady state while the other half has initial conditions that lead to the higher vegetated state. The two boundary points had the same value as their respective sides initial conditions and were set to be constant for all time. These initial conditions lead to a picture just like 2

Figure 2: A graph describing a liquid-vapour equilibrium across a one dimensional planar interface fluid. Where ρ is the density and is a function of the distance z which is perpendicular to the interface of the densities. Taken from (Widom 1985)



Using these initial conditions, we allowed the lattice to evolve in time sufficient enough for the interface between the two steady states to move in either direction. We then varied the bifurcation parameter to see if there was a change in the speed of propagation or a reversal in which state is more stable. We chose the bifurcation parameter c at different points of the range 18-26 because this is where multiple steady states are available to our system. See the section on bifurcation for clearer explanation. We measured the velocity by evolving our lattice for time t and then visually measuring how far the inter-

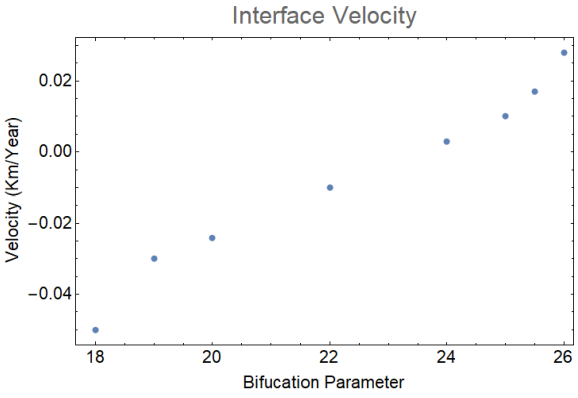
Table 1: This table lists parameter values as they appear in the program and in their fundamental units as related to Δx and Δt. Those with stochastic noise are indicated in the list. For explanation of the σ look at the stochasticity section

Parameter	Value in program	Value as related to Δx and Δt	Stochastic Noise
D	.03	$8.22m^2$/Days	NA
r	10	27.391/Days	$\sigma_r = 1$
c	$18 - 24$	$49.32 - 71.23$ vegetation-Biomass/Days	$\sigma_c = 20$
Δx	.1	$10m$	NA
Δt	.001	.001 years	NA
V_0	1	1 vegetation-Biomass	NA
V_c	10	10 vegetation-Biomass	NA

face moved in that time. Then we divided the distance traveled by the time we let the lattice evolve.

We notice key points about this graph mainly somewhere around $c = 23.5$. We see that the relative stability of the phases switch from the high vegetative state being more stable to the lower vegetative state becoming more stable. This indicates that if we can change the bifurcation parameter to be below $c = 23.5$ we may be able to halt or even revert the propagation of a lower vegetative state. This is in contrast to what has been seen before where c would have to be pushed back to the bifurcation point for it to revert to a high vegetated state. We also notice that, as we approach the bifurcation points, we see a steeper slope which is expected because at bifurcation one of the states is no longer a steady state so it cannot be stable.

Figure 3: Graph showing how interface velocity in km/Y changes as a function of our bifurcation parameter c in 1-D with no noise. A positive velocity value indicates the lower vegetated state propagating and overtaking the higher vegetated state. A negative velocity represents the opposite.



ii. Interface Velocity Related to a Potential

We now want to find a way to think about this propagation of the interface to get a better understanding of how it works in this system. Since this interface is moving with some velocity u, we can make a Galilean transformation $z = x - ut$. In other words $V(x,t) \to V(z)$. From this we see that $\frac{\partial}{\partial x} = \frac{\partial z}{\partial x}\frac{\partial}{\partial z}$ and similar in t so $\frac{\partial}{\partial t} = \frac{\partial z}{\partial t}\frac{\partial}{\partial z}$. So we can rewrite our equation as

$$-u\frac{dV}{dz} = D\frac{d^2V}{dz^2} + f(V)$$
$$0 = D\frac{d^2V}{dz^2} + u\frac{dV}{dz} + f(V)$$

Where $f(V) = rV(z)(1 - \frac{V(z)}{V_c})$ $- c\frac{V(z)^2}{V(z)^2+V_0^2}$. We now multiply this equation by $\frac{\partial V}{\partial z}\partial z$ and integrate it from $-\infty$ to ∞. If we look term by term then we see that

$$D\int_{-\infty}^{\infty} dz \frac{dV}{dz}\frac{d^2V}{dz^2}$$

If we write $\frac{d^2V}{dz^2}$ as $\frac{d}{dz}\frac{dV}{dz}$ then we see that this terms becomes

$$D\int_{-\infty}^{\infty} dz \frac{d}{dz}(\frac{dV}{dz})^2$$

Then by the fundamental theorem of calculus this becomes

$$D(\frac{dV}{dz})^2|_{-\infty}^{\infty}$$

We assume the slope of this interface goes to zero at infinity so this means this term is identically zero.

If we assume the last term comes from a potential, then we have relationship $f(V) = -\frac{dU}{dV}$. So we now have

$$\begin{aligned}\int_{-\infty}^{\infty} dz f(V)\frac{dV}{dz} &= -\int_{-\infty}^{\infty} dz \frac{dU}{dV}\frac{dV}{dz} \\ &= \int_{-\infty}^{\infty} -dU \\ &= U(-\infty) - U(\infty)\end{aligned}$$

Finally, we have our second term.

$$\int_{-\infty}^{\infty} (\frac{dV}{dz})^2 dz$$

we know that

$$\int_{-\infty}^{\infty} (\frac{dV}{dz})^2 dz$$

is a positive definite number so we call it K. So we are left with

$$uK = U(\infty) - U(-\infty)$$

If we then say at $U(\infty)$ it is the highly vegetated state, or U_{veg}, and $U(-\infty)$ is barren, or U_b, then to have a positive velocity we say the potential of the barren state U_b is less than U_{veg} and the lower vegetative state or barren state overtakes the lattice and if $u < 0$ then U_b is greater than U_{veg} so the highly vegetated state overtakes the lattice.

The main assumption we made here is that $\frac{\partial f_i}{\partial V_j} = \frac{\partial f_j}{\partial V_i}$, which is valid since we have a one dimensional case so by the fundamental theorem of calculus this is justified.
This mathematical manipulation allows us to view this phenomenon in a more familiar light. Establishing that there is a potential which is responsible for this propagation and is proportional to velocity also allows one to ask the question in the future 'what is the functional form of this potential?'

iii. Stochasticity

After intial investegation into May's idea people wanted to start adding spatial inhomogeneity to simulate ecological inhomogeneity, to do this we use stochastic noise. Stochastic noise is defined such that, for every point in time, it is a random value but is also continuous in time. We use this to simulate continuous randomness in the ecosystems that we cannot account for in other ways. Here we use delta correlated noise with mean zero such that a stochastic variable $\eta(t)$ has time averages of $\langle\eta(t)\rangle = 0$ and $\langle\eta(t')\eta(t')\rangle = \sigma^2\delta(t-t')$. We note here that due to the continuum limit, we have a delta function which has units of $[1/t]$. So σ has units of $[\eta(t)]/\sqrt{[t]}$ where $[\eta(t)]$ represents the units of $\eta(t)$ and so on.
For our purposes, we will have stochastic noise in our c parameter and r parameter which means our equation becomes.

$$\begin{aligned}\frac{\partial V}{\partial t} = D\nabla^2 V + (r+\eta_1(t))V(1-\frac{V}{V_c}) \\ -(c+\eta_2(t))\frac{V^2}{V^2+V_0^2}\end{aligned}$$

For simplicity's sake, I will just show how we handle the noise in r. We essentially have a differential equation that says $\dot{V} = rf(V(t)) + \eta_1(t)f(V(t))$. We then again go to the definition of a derivative. However, if we try to simply solve for $V(t+\Delta t)$, then we encounter a problem. How does $\eta_1(t)$ change between t and Δt? So

instead we write

$$\dot{V} = rf(V(t)) + \eta_1(t)f(V(t))$$
$$V(t) = \int_t^{t+\Delta t} rf(V(s))ds + \int_t^{t+\Delta t} f(V(s))\eta_1(s)ds$$

We can approximate the first integral as $rf(V(t))\Delta t$, but the second integral isn't as easy to deal with. Thankfully, since this integral is an infinite sum of a probability distribution, the central limit theorem guarantees that this is a Gaussian distribution. We have already said that $\langle \eta(t) \rangle = 0$, so the mean of our Gaussian is zero and our variance $\langle \eta(t)^2 \rangle - \langle \eta(t) \rangle^2 = \sigma^2\sqrt{t}$ and our Gaussian distribution has a standard deviation of $\sigma\sqrt{t}$. So in our update scheme we can use a Gaussian random number generator times $\sigma\sqrt{\Delta t}$, which replicates this behavior. Looking at table 1, this σ is what is being refereed to in the table.

iv. Interface Velocity with Stochastic Noise

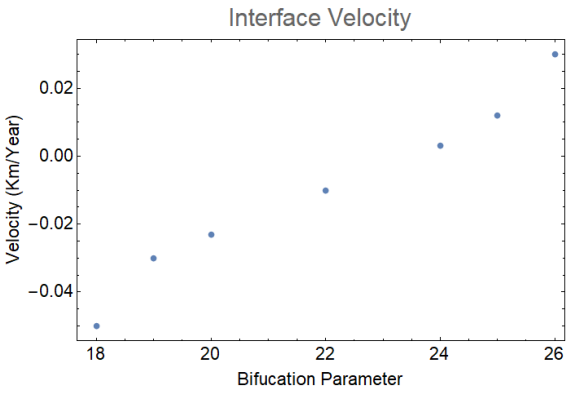
We then want to see if these ideas hold up when we add stochastic noise in our bifurcation parameter c and in our r parameter. This stochastic noise adds a layer of realism to this system. Our equation now becomes

$$\frac{\partial V}{\partial t} = D\nabla^2 V + (r + \eta_1(t))V(1 - \frac{V}{V_c}) - (c + \eta_2(t))\frac{V^2}{V^2 + V_0^2}$$

We treat this noise as described above in the stochasticity section. We set up our lattice just like we did in the section with no noise. We have half of the lattice set such that it will become the lower vegetated state and the other half will become the highly vegetated state. We then again allow it to evolve in time, this time with stochastic noise, and see how far the interface between the two phases travels. We then record the velocity and do this for the same values of c as before.

We notice that the general form of this graph is still roughly the same. Near by the bifurcation points we still have larger slopes and the zero is still roughly around $c = 23.5$. This indicates that small amounts of stochastic noise doesn't have an extraordinarily large effect on the interface velocity. This also continues to allow us to think about this velocity as derived from a potential even when we include stochastic noise. All conclusions we made without stochastic noise are still true once we add it.

Figure 4: Graph showing how interface velocity in km/Y changes as a function of our bifurcation parameter c in 1-D with stochastic noise. A positive velocity value indicates the lower vegetated state propagating and overtaking the higher vegetated state. A negative velocity represents the opposite.



IV. 2-D

i. Motivation

We then move into the 2-D realm. We do this for multiple reasons. One, we add another level of realism because we can think of a forest ecosystem as a plane in the xy plane with $V(x, y)$ the vegetation on the z axis. Two, we want to study the same fundamentals of the 1-D system to see if similar phenomenons are present. Three, to try things not possible in a one dimensional lattice such as setting up patches of different vegetative phases and seeing their evolution. The last reason is this has yet to be done in other literature so any observations of a 2-D lattice will be new.

ii. 2-D Interface No Noise

We want to establish a baseline for how our 2-D lattice acts and observe the relative stability of the different vegetated phases as we did in one dimension. This will better inform our observations when doing other investigations into our 2-D lattice's behavior. Our 2-D lattice is described by the following equation

$$\frac{\partial V(x,y,t)}{\partial t} = D\nabla^2 V(x,y,t) + rV(x,y,t)(1 - \frac{V(x,y,t)}{V_c}) - c\frac{V(x,y,t)^2}{V(x,y,t)^2 + V_0^2}$$

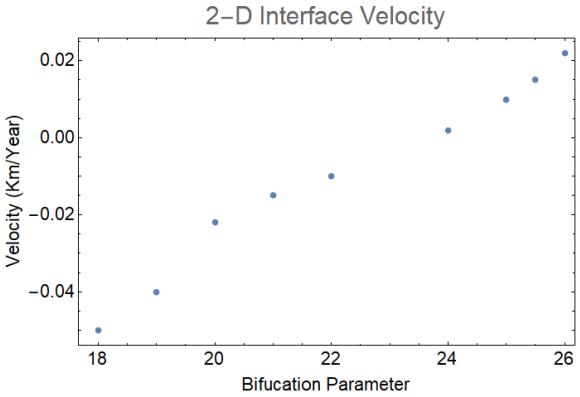
The only difference in the equation is now we have a y component. The total lattice is now size $N \times N$ in place of our singular N in one dimension with a boundary. We set up the interface like before, splitting the lattice in half. The first half which is set to the low vegetated state runs from $x = 0$ to $x = N/2$ and y runs from zero to N. The initial conditions are set up such that this side goes to the lower vegetated state according the values in the bifurcation diagram. The boundaries are set to be constant with the same value as the initial condition. Then the other half, which ranges from $x = N/2$ to $x = N$, is set up with initial conditions such that it reaches the high vegetated state with the boundaries set to the same values as the initial condition and then set to a constant. With this set up, any fixed coordinate of x will yield the same value of $V(x, y)$ for all y coordinates. We then find the spatial average of $V(x, y)$ over all y for different x. This then creates an interface identical to the 1-D case so we can use the same methods to observe motion.

We see that all the same behavior is carried through from the one dimensional case. This shows that, in a more realistic model, if the bifurcation parameter is passed 23.5 then, on time scales of 20 years, we can see vegetation loss on the order of of Km^2.

We make a parenthetical remark about time scales. There is one time scale on which the system approaches the steady-state fixed-point value in mean-field theory (without diffusion). A nave estimate or a numerical simulation shows that this is of the order of a few days. On the other hand, time scale on which interface motion occurs (See Table 2) and other diffusion-dependent phenomena (such as growth or erosion of patches in the next section) occur is considerably longer, of the order of years. This underlines the need for including spatial degrees of freedom when determining the time scales on which significant ecological changes occur.

This allows us to ask more interesting questions since this phenomenon is still true in 2-D. Specifically, if there is a critical size for the patches before we observe growth? Can patches combine and support even if one is below the critical size? Can we stop this growth without changing the bifurcation parameter everywhere? But first we must try and observe this phenomenon with stochastic noise.

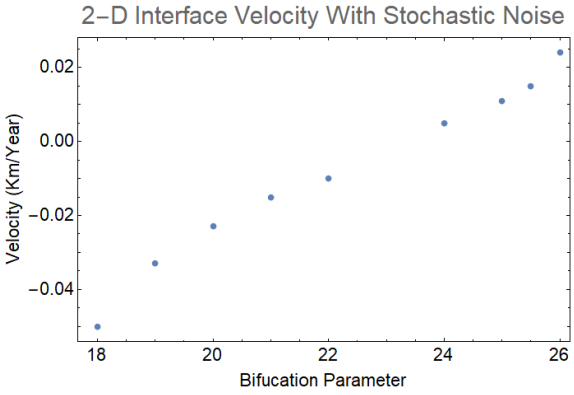
Figure 5: Graph showing how interface velocity in km/Y changes as a function of our bifurcation parameter c in 2-D with no noise. A positive velocity value indicates the lower vegetated state propagating and overtaking the higher vegetated state. A negative velocity represents the opposite.



iii. 2-D Interface Velocity With Stochastic Noise

We again want to add another level of realism to our system so we add stochastic noise to the same parameters as before, r and c. This stochastic noise is again delta correlated with mean zero. The lattice is set up the same as above with half of the lattice set to be in the

Figure 6: Graph showing how interface velocity in km/Y changes as a function of our bifurcation parameter c in 2-D with stochastic noise. A positive velocity value indicates the lower vegetated state propagating and overtaking the higher vegetated state. A negative velocity represents the opposite.



lower vegetated state and the other half to be the highly vegetated state. We plot the average value of x when averaged over all y and then see how it propagates in time.

We see very similar behavior as before, which indicates the noise does not drastically effect this behavior. The one thing we note is that around $c = 23.5$, it is more linear until it approaches the bifurcation points. This makes sense, as the noise should affect how diffusion spreads this slightly and slow it down. But, once we near bifurcation, the noise can push points past bifurcation, which is why we still have values very close to the 2-D no noise at the bifurcation points.

iv. Nucleation Size

The next question we looked to address was if there is a critical size where we can start to witness growth. This was motivated by nucleation Theory (Maris 2006). However, this system does not have temperature or free energy. There exists a critical radius such that before this radius the low vegetated state becomes overtaken by the high vegetated state and after the radius the low vegetated state expands into the high vegetated state. This is is pertinent because, if given accurate data, we would have an idea when outside intervention in the ecosystem is necessary to prevent the spread of a low vegetated patch.

To find this behavior, we choose a value of c such that we expect the low vegetated state to be more stable and propagate. We then set the initial conditions of the lattice to be the following. First, the entirety is given initial conditions such that it should reach the highly vegetated state. Then, centered in the middle at $(N/2, N/2)$, we create a patch of size $n \times n$ with initial conditions that lead to the lower vegetated state. We also have our overall lattice size set to $1 * 1Km^2$. With this all in place, we then let it evolve in time and mark the time at which the growth of the lower vegetated patch overtakes the lattice. 'Overtakes' operational definition here is when the lower vegetated patch reaches the boundary of the total lattice.

From the table we can see that at a patch $50*50m^2$ with $c = 24.5$, we have the highly vegetated state overtaking the patch but at $60*60m^2$, the low vegetated patch actually slowly overtakes the lattice. Since our spatial step is 10 meters, we consider the critical radius to be $50m$. However, this is only for when $c = 24.5$. If you look again at the table, we see if c is changed to 25 that $50 * 50m^2$ actually overtakes the lattice. So we have identified that the critical radius is a function of c and can be found through simulations if c is available.

v. Multiple Small Patch Effects

Now that we understand there is a critical size, the next thing we want to investigate is if this is true for multiple patches spread apart. These two states of high vegetation and low vegetation have been seen to exist in real ecosystems such as semi arid desert regions or bogs (Reitkerk et al 2004). The existence of these two phases interspersed within a singular ecosystem prompts investigation into the relative stability of these phases and motivates the investigation into interface velocity of these different phases and investigation of effects of patch size and if

Table 2: This table represents different trials detailed above. The size column refers to the size of the low vegetated patch that was created at the center of the lattice. The time to overtake column refers to the time for the patch to eventually grow to reach the boundary of the lattice. The error on this measurement is due to general uncertainty of when it first comes into contact with the boundary. The units on c are the same as given in table 1. A negative time to overtake represents that the lower vegetated patch was engulfed by the higher vegetated state.

Size	Time to Overtake	c
$160 * 160m^2$	62 ± 2 years	24.5
$120 * 120m^2$	67 ± 2 years	24.5
$100 * 100m^2$	75 ± 3 years	24.5
$80 * 80m^2$	80 ± 3 years	24.5
$60 * 60m^2$	84 ± 3 years	24.5
$50 * 50m^2$	-7.5 ± 0.5 years	24.5
$50 * 50m^2$	-4.5 ± 0.5 years	24
$50 * 50m^2$	45 ± 2 years	25
$40 * 40m^2$	-3.5 ± 0.5 years	24.5

multiple patches lead to different behavior. If we identify multiple patches that are low vegetative, which has been shown to be apparent in semi arid regions (Rietkirk et al 2002), then we want to know if we need to only worry about the critical radius to identify if these patches will propagate or if these patches can feed into one another.

To answer this question, we set up a situation with four patches of low vegetation on the lattice, where the rest of the lattice is set to the high vegetated state with $c = 24.5$ everywhere. These patches are centered at the following locations: patch 1 $(200m, 200m)$, patch 2 $(340m, 240m)$, patch 3 $(810m, 810m)$, patch 4 $(530m, 430m)$ and are sizes: patch 1 $70 * 70m^2$, patch 2 $40 * 40m^2$, patch 3 $90 * 90m^2$, patch 4 $(100m^2, 100m^2)$. With this setup, a random point on the lattice is still more likely to be highly vegetated then not. From this setup we hope to observe two things. One, how quickly does this overtake the lattice with these patches and does patch 2 become overtaken by the high vegetation because it is smaller than the critical radius for when $c = 24.5$, or will it combine with patch 1 and continue to cause damage by propagating the low vegetative state.

The result was there was an increase in the speed in which overtaking the lattice happens and we observed patch 1 and patch 2 combining to keep propagating the lower vegetated state. This indicates that, to stop propagation of the low vegetative state, we should worry about patches bigger than the critical radius and patches that are near others. Future investigations in this would include exact numeric measuring of these effects and an attempt to identify a critical range based on relative size of the patches and distance between them.

vi. Stopping the Spread of Low Vegetation

Our final investigation was on how to stop the spread of low vegetated patches. This would allow one who has observed conditions that lead to a regime shift to halt it. To test ways to stop the spread of the low vegetated state, we set up an interface with stochastic noise the same way we did to measure the velocity of propagation. Half the lattice is set to the high vegetated state and half the lattice is the lower vegetated state. We set $c = 25$, which would allow the low vegetated state to propagate.

It was found that a line along the interface where $c = 18$ stops the propagation completely. How-

ever, this raises the question of what is the least amount we must do to stop the propagation. After some testing, it was seen that a line that has 4 points of $c = 18$ then 6 points where $c = 25$ suffices to stop the propagation of the lattice. But if we change to 3 and a gap of 7, then the low vegetative state still spreads.
More work can be done to see how to slow or even reverse the effect and propagate the high vegetated state and gather numeric values of each situation.

V. Conclusion

We have identified many new behaviors within a system that describes real world grazing ecosystems. Each state available to the system has potential that can drive it towards the other state depending on the bifurcation parameter c. This can drive a lattice in 2-D to a lower vegetated steady state if unchecked on the order of 10s of years. This highlights the need for spatial interpretations of stability as opposed to mean value models as the end behavior and time scales at what they reach are drastically different. Patches of low vegetation can combine and destabilize a highly vegetated state. Finally, there is a way to pin this undesirable growth with intervention that doesn't require a complete change of parameters over all space.
This opens many avenues for future work, the most obvious being refinement of these results. We must acquire more extensive data that pinning can stop the growth of bare patches or if we can use our method of pinning growth to stop the combination of propagating low vegetative patches. However, once this is completed, we want to treat consumption and growth rate as other as coupled dynamical variables with vegetation. This will lead to interesting theoretical issues as there is no reason in general for there to be a potential from which the terms that drive the dynamics can be derived. Our analysis in 1d cannot be done simply and one must investigate which effects will be qualitatively the same.

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